VOLUME ENTROPY AND LENGTHS OF HOMOTOPICALLY INDEPENDENT LOOPS

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ABSTRACT. This paper presents an inequality for closed Riemannian manifolds involving the volume entropy and the set of lengths of any family of homotopically independent geodesic loops based at the same point. This inequality implies a curvature free collar theorem, and is an analogous of the main theorem of [ACCS96] for variable curvature Riemannian manifolds. Its proof is rather straightforward once we know the work by Lim [Lim08] on volume entropy for graphs.

For a closed Riemannian manifold M we denote by $\widetilde{M}$ its Riemannian universal cover. The *volume entropy* (also known as *asymptotic volume*) of M is defined as the following quantity:

$$h_{\mathrm{vol}}(M) := \lim_{R\to\infty} \frac{\log \mathrm{vol}\, B(\widetilde{x}, R)}{R}.$$

Here $B(\widetilde{x}, R)$ denotes the metric ball of radius R around some point $\widetilde{x}$ in $\widetilde{M}$ and vol the Riemannian volume. This limit always exists and does not depend on the chosen point $\widetilde{x}$ (see [Man79]). This asymptotic invariant describes the exponential growth rate of the volume of balls in the universal cover and highly depends on how the fundamental group and the geometry of the manifold interplay. Indeed the volume entropy can also be defined as the exponential growth rate of the number of homotopy classes with bounded length thanks to the classical identity:

$$h_{\mathrm{vol}}(M) = \lim_{R\to\infty} \frac{\log \#\{\gamma \in \pi_1 M \mid d(\widetilde{x}, \gamma\cdot\widetilde{x}) \leqslant R\}}{R}. \tag{1}$$

Note that for an element $\gamma \in \pi_1 M$ the distance $d(\widetilde{x}, \gamma\cdot\widetilde{x})$ coincides with the length $\ell(c)$ of a shortest geodesic loop c in the class γ and based at x (the projection of $\widetilde{x}$ on M by the covering map). In view of this equality it seems reasonable to look for explicit relations

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between $h_{\mathrm{vol}}(M)$ and the length of some subfamily of these geodesic loops. That was the starting point of our investigation. Remember that n loops are said *homotopically independent* if their homotopy classes generate a free subgroup of rank n. Here is our result.

Theorem 1. *Let M be a closed Riemannian manifold, $x \in M$ and $n \geqslant 2$. Assume that there exists a family $c_1, \dots, c_n$ of homotopically independent loops based at x.*

Then the following inequality holds true:

$$\sum_{k=1}^{n} \frac{1}{1 + e^{\ell(c_k) \cdot h_{vol}(M)}} \leqslant \frac{1}{2}.$$

This inequality has some similarity with the Anderson-Canary-Culler-Shalen inequality [ACCS96, main theorem] which states that, if Γ is a purely loxodromic, topologically tame and discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$, freely generated by $c_1, \cdots, c_n \in \pi_1(\mathbb{H}^3/\Gamma, x)$, then

$$\sum_{k=1}^{n} \frac{1}{1 + e^{\ell(c_k)}} \leqslant \frac{1}{2}.$$

Theorem 1 applies in particular for free subgroups of a fundamental group of a hyperbolic 3-manifold (such as Schottky groups) and we recover the above inequality in this particular case, although with a worst constant, since the volume entropy of a hyperbolic 3-manifold equals 2.

Nevertheless, Theorem 1 is optimal in the general setting of arbitrary Riemannian metrics in the sense that, for any $n \geqslant 2$, there exists a sequence of Riemannian metrics $\{g_i\}_{i \in \mathbb{N}}$ on the connected sum X of n copies of $S^1 \# S^2$, a point x on X and a family of homotopically independent loops $c_1, \dots, c_n$ based at x such that

$$\lim_{i \to \infty} \sum_{k=1}^{n} \frac{1}{1 + e^{\ell_{g_i}(c_k) \cdot h_{vol}(X, g_i)}} = \frac{1}{2}.$$

See Remark 5 for more details.

As a consequence of our theorem we directly get the following.

Corollary 2. *Fix $h > 0$. Let M be a closed Riemannian manifold with volume entropy h. Suppose that c_1 and c_2 are two homotopically independent loops based at x.*

Then

$$\ell(c_2) \geqslant \frac{1}{h} \log\left(\frac{4}{h\ell(c_1)}\right) + o(1)$$

for $\ell(c_1)$ sufficiently close to 0.

So if the volume entropy is keeped fixed while the shortest length shrinks to zero, then the length of the largest loop blows up and we control the rate of explosion. It partially recovers (albeit with a worst multiplicative constant) and also generalizes to free curvature metrics the classical consequence of the collar theorem [Bus92, Corollary 4.1.2] that given a closed hyperbolic surface S and two simple closed geodesics c_1 and c_2 intersecting each other, then the following sharp inequality is satisfied:

$$\sinh\left(\frac{\ell(c_1)}{2}\right)\sinh\left(\frac{\ell(c_2)}{2}\right) > 1.$$

Indeed this inequality admits the following expansion

$$\ell(c_2) \geqslant 2\log\left(\frac{4}{\ell(c_1)}\right) + o(1)$$

for $\ell(c_1) \to 0$ while $h_{\mathrm{vol}}(S) = 1$.

After showing Corollary 2, we discovered that a curvature free analog of collar theorem was already shown in [BCGS17, Lemma 7.12] where they obtain that

$$\ell(c_2) > \frac{1}{h}\log\left(\frac{1}{h\ell(c_1)}\right)$$

under the same assumptions. Our corollary slightly improves their result for small values of $\ell(c_1)$, but most importantly Theorem 1 relates it to a more general inequality. Compare also with [Cer14, Theorem 1.2].

For largest families of homotopically independent loops we can also bound from below the length of the largest one in terms of the previous ones.

Corollary 3. *Fix $h > 0$ and $n \geqslant 3$. Let M be a closed Riemannian manifold with volume entropy h. Suppose that $c_1, \ldots, c_n$ are n homotopically independent loops based at x and ordered by increasing length: $\ell(c_1) \leqslant \ldots \leqslant \ell(c_n)$.*

Then

$$\ell(c_n) \geqslant \frac{-1}{h}\log\left(\frac{h\ell(c_1)}{4} - \sum_{k=2}^{n-1} e^{-h\ell(c_k)}\right) + o(1)$$

for $\ell(c_1)$ sufficiently close to 0.

To illustrate which information provides this inequality, observe that if the second length is ε-closed to the lower bound in Corollary 2, then the third length blows up at a speed at least $-\log\varepsilon/h$.

We now prove Theorem 1.

Let M be a closed Riemannian manifold, $x \in M$ and $n \geqslant 2$. Assume that $c_1, \dots, c_n$ is a family of homotopically independent loops based at x. We denote by $a_i := \ell(c_i)$ their respective lengths and suppose that they are ordered as follows: $a_1 \leqslant \dots \leqslant a_n$.

On the universal cover $\widetilde{M}$ fix a lifted point $\tilde{x}$ of x and consider the metric graph $\widetilde{G}$ defined as follows. The vertices of $\widetilde{G}$ are in one-to-one correspondence with points $\{\gamma \cdot \tilde{x} \mid \gamma \in \langle c_1, \dots, c_n \rangle \simeq \mathbb{F}_n \subset \pi_1 M\}$, and two vertices $\widetilde{y}$ and $\widetilde{z}$ are connected through an edge of length a_i if and only if $\widetilde{z} = c_i^{\pm 1} \cdot \widetilde{y}$. This graph is an infinite tree of valence $2n$ and is the universal cover of the metric graph denoted by $G_{a_1,\dots,a_n}$ defined as the wedge product of n circles of respective lengths $a_1, \dots, a_n$. It is easy to check that

$$\#\{\tilde{v} \in V(\tilde{G}) \mid d_{\tilde{G}}(\tilde{x}, \tilde{v}) \leqslant R\} \leq \#\{\gamma \cdot \tilde{x} \mid d(\tilde{x}, \gamma \cdot \tilde{x}) \leqslant R\}.$$

Equality (1) thus implies that

$$h_{vol}(M) \geqslant h_{vol}(G_{a_1,\dots,a_n}).$$

Then the announced inequality

$$\sum_{i=1}^{n} \frac{1}{1 + e^{a_i \cdot h_{vol}(M)}} \leqslant \frac{1}{2}$$

is a straightforward consequence of the following result for graphs.

Lemma 4. *The volume entropy $h := h_{vol}(G_{a_1,\dots,a_n})$ satisfies the following equality:*

$$\sum_{i=1}^{n} \frac{1}{1 + e^{h a_i}} = \frac{1}{2}.$$

Proof. According to [Lim08, Theorem 4] we know that h is the only positive real number such that the following linear system of equations with unkowns x_i

$$\begin{cases} x_1 = x_1 e^{-h a_1} + 2x_2 e^{-h a_2} + \dots + 2x_n e^{-h a_n} \\ x_2 = 2x_1 e^{-h a_1} + x_2 e^{-h a_2} + \dots + 2x_n e^{-h a_n} \\ \qquad\qquad \vdots \\ x_n = 2x_1 e^{-h a_1} + 2x_2 e^{-h a_2} + \dots + x_n e^{-h a_n} \end{cases}$$

has a solution with $x_i > 0$ for $i = 1, \dots, n$. So take such a solution $(x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n$. By summing all the equations we see that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (2n-1) e^{-h a_i} x_i,$$

and by substracting any two different lines (L_i) and (L_j) we get that

$$(1 + e^{-h a_i}) x_i = (1 + e^{-h a_j}) x_j.$$

So

$$\sum_{i=1}^{n} \frac{1-(2n-1)e^{-ha_i}}{1+e^{-ha_i}} = 0$$

which is equivalent to the inequality

$$\sum_{i=1}^{n} \frac{1}{1+e^{ha_i}} = \frac{1}{2}.$$

□

Remark 5. This result is optimal in the sense that, for any $n \geqslant 2$, there exists a sequence of Riemannian metrics $\{g_i\}_{i\in\mathbb{N}}$ on the connected sum X of n copies of $S^1 \times S^2$, a point x on X and a family of homotopically independent loops $c_1, \ldots, c_n$ based at x such that

$$\lim_{i\to\infty} \sum_{k=1}^{n} \frac{1}{1+e^{\ell_{g_i}(c_k)\cdot h_{vol}(X,g_i)}} = \frac{1}{2}.$$

The construction of the sequence of metrics $\{g_i\}_{i\in\mathbb{N}}$ can be easily obtained by slightly modifying the simplicial Riemannian metric defined on the wedge product of n copies of $S^1 \times S^2$ as follows. For $k = 1, \ldots, n$ consider on each copy $(S^1 \times S^2)_k$ the metric product $a_k^2 dt \otimes ds$ where dt denotes the standard Riemannian metric on S^1 of length 1 and ds the standard Riemannian metric on S^2 of area 4π. Then the simplicial Riemannian metric g it induces on the wedge product $\vee_{k=1}^{n}(S^1 \times S^2)_k$ has the following property. If x denotes the common point to all factors, any minimal (in its homotopical class) geodesic loop γ based at x decomposes as a unique concatenation $\alpha_1 \star \ldots \star \alpha_N$ where each α_j is a minimal geodesic loop in some factor $(S^1 \times S^2)_{k_j}$ and whose class is the p_j-iterated for some $p_j \in \mathbb{Z} \setminus \{0\}$ of a generator of the corresponding fundamental group. It is thus straighforward to see that $\ell(\gamma) = \sum_{j=1}^{N} |p_j| \cdot a_{k_j}$ from which we deduce that the volume entropy of $(\vee_{k=1}^{n}(S^1 \times S^2)_k, g)$ is equal to the volume entropy of $G_{a_1,\ldots,a_n}$. Now observe that we can choose for each $k = 1, \ldots, n$ as c_k the unique minimal geodesic loop based at x and contained in $(S^1 \times S^2)_k$ that corresponds to one of the two generators of the fundamental group of this factor. In particular

$$\sum_{k=1}^{n} \frac{1}{1+e^{\ell_g(c_k)\cdot h_{vol}(\vee_{k=1}^{n}(S^1\times S^2)_k,g)}} = \frac{1}{2}.$$

Finally we construct the sequence of metrics g_i to smooth out the base point x. Indeed we see the metric g as a singular metric on the connected sum $\#_{k=1}^{n}(S^1 \times S^2)_k$ and we approximate g in the C^0-topology by smooth Riemannian metrics. The conclusion follows as both length and volume entropy are continuous maps for this topology.

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